

# Inconsistency of a dissipative contribution to the mass flux in hydrodynamics

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## Abstract

The possibility of dissipative contributions to the mass flux is considered in detail. A general, thermodynamically consistent framework is developed to obtain such terms, the compatibility of which with general principles is then checked—including Galilean invariance, the possibility of steady rigid rotation and uniform center-of-mass motion, the existence of a locally conserved angular momentum, and material objectivity. All previously discussed scenarios of dissipative mass fluxes are found to be ruled out by some combinations of these principles, but not a new one that includes a smoothed velocity field  $\bar{v}$ . However, this field  $\bar{v}$  is nonlocal and leads to unacceptable consequences in specific situations. Hence we can state with confidence that a dissipative contribution to the mass flux is not possible.

## I. THE PROBLEM AND ITS HISTORY

The equality of mass flux and momentum density is widely accepted as a statement of plausibility in hydrodynamics, supported by countless experiments. However, in a classical paper, Dzyaloshinskii and Volovick [1] proposed the inclusion of a dissipative mass flux into the hydrodynamic equations. Starting from a modified kinetic theory, Klimontovich [2] arrived at the same suggestion. More recently, a dissipative contribution to the mass flux was re-introduced and forcefully promoted by Brenner [3, 4], whose work stimulated significant interest and controversy in the physics and fluid dynamics communities.

In the early work of Brenner [3], the mass flux and the momentum density are taken to be equal, and only the volume velocity, used in Newton's expression for the stress tensor and in no-slip boundary conditions, differs by a dissipative term from the mass velocity (see also Brenner's most recent work [5]). Brenner's idea has been expanded in [6], where not only Newton's expression for the stress tensor but also Fourier's law for the heat flux has been modified. These methods yield some elements, e.g. thermal stresses, that are known in kinetic theory of gases as rarefaction effects which only play a role beyond the hydrodynamic regime, that is, at finite Knudsen numbers.

Indeed, extensions of hydrodynamics to the finite Knudsen numbers are widely discussed within kinetic theory (e.g., Burnett equations, R13 equations, extended thermodynamics [7]); none of these includes dissipative contributions to mass flux, and all reduce to classical hydrodynamics in the limit of small Knudsen numbers. The modifications to hydrodynamics in [3, 6] are not able to produce the whole wealth of rarefaction effects known in kinetic theory. Moreover, to our knowledge, they were never tested successfully against solutions of the Boltzmann equation, or experiments.

Greenshields and Reese [8, 9] used Brenner's equations of both scenarios [3, 4] to describe experimental findings for the structure of shock waves. In their work, the modified hydrodynamic equations are actually considered as the simplest way of introducing finite Knudsen number effects [7]. However, the equations should be subjected to a much wider array of test problems at small and finite Knudsen numbers, before one can claim that they give meaningful results. Because we are interested in hydrodynamic theories introducing a difference between mass flux and momentum density, however, we do not discuss the ideas of [3, 6] any further.

In an unpublished precursor version of [3], dated October 30, 2002, Brenner had actually proposed a difference between mass flux and momentum density, which is the topic of the present work. He later returned to this original idea in [4], with a specific recommendation for the cross coupling of density and temperature effects based on his “incompressibility hypothesis.”

Brenner’s work motivated thorough investigations on the thermodynamic admissibility of a dissipative contribution to the mass flux [10, 11, 12, 13, 14], which demonstrated that the idea fits naturally into the GENERIC framework (“general equation for the nonequilibrium reversible-irreversible coupling”) and into standard linear irreversible thermodynamics. However, these investigations focused entirely on nonequilibrium thermodynamics and neglected other, equally important considerations, such as the local conservation of angular momentum. Such additional criteria were considered earlier in [15] and corroborated the original (not rigorously justified) statement of Landau and Lifshitz that mass flux and momentum density must be equal (see footnote at the end of Sec. 49 of [16]). A concise summary of the current state of the discussion can be found in the comment [17] to the letter [14], and in the reply [18] to that comment.

To obtain a more complete picture, we here first develop a general thermodynamically consistent scenario of hydrodynamics with dissipative mass flux and then apply the following additional criteria: Galilean invariance, possibility of steady rigid fluid rotation, existence of a locally conserved angular momentum, and consistency with uniform center-of-mass motion. To keep the present analysis as simple as possible, we restrict ourselves to single component fluids, we assume the absence of external forces, and we use a vanishing bulk viscosity. All these limitations can be removed without changing the arguments or conclusions in any essential way.

In the following we shall show that a consistent system of hydrodynamic equations with a difference between mass flux and momentum density is only possible if an additional velocity, denoted as  $\bar{\mathbf{v}}$ , is introduced. While this velocity first arises as a modeling possibility—a free parameter—within the GENERIC framework, it turns out that the criteria listed in the previous paragraph can only be fulfilled for non-vanishing and homogeneous  $\bar{\mathbf{v}}$ . However, we shall argue that a homogeneous velocity with these properties is non-physical. We conclude that dissipative contributions to mass flux are forbidden by the intricate interplay of the various physical requirements. Previous discussions did not consider the full breadth of

physical criteria, and thus could not completely exclude dissipative mass flow.

## II. DEVELOPMENT OF MODIFIED HYDRODYNAMIC EQUATIONS

To provide a reliable starting point for our discussion, we first introduce a generalization of a previously developed thermodynamically consistent scenario for hydrodynamic equations with a dissipative contribution to the mass flux within the GENERIC framework of nonequilibrium thermodynamics. Because we are concerned with the mass flux and the momentum density, the key equations are the continuity equation for the mass density  $\rho$  and the equation of motion for the momentum density  $\mathbf{M}$ . To elucidate thermodynamic aspects and to identify dissipative fluxes unambiguously, we add the balance equation for the entropy density  $s$  to obtain a complete set of hydrodynamic equations, so that our final set of equations is for the variables  $x' = (\rho, \mathbf{M}, s)$ . For the formulation of the friction matrix, however, it is more convenient to work with the variables  $x = (\rho, \mathbf{M}, \epsilon)$ , where  $\epsilon$  is the internal energy density.

For the formulation of classical hydrodynamics within the GENERIC framework of nonequilibrium thermodynamics, we refer the reader to Section 2.2 of [10]. The generators energy and entropy are given there, together with the Poisson and friction matrices turning the gradients of energy and entropy into the reversible and irreversible contributions to time evolution, respectively. We here restrict ourselves to the formulation of a dissipative contribution to the mass flux, which is achieved through an additional contribution to the friction matrix. Compared to the diffusive contribution  $M^{\text{diff}}$  to the friction matrix given in Eq. (2.77) of [10], we here consider a more general choice of a factorized form of rank unity,

$$M^{\text{diff}} = C \cdot D' C^T, \quad (1)$$

where  $D' \geq 0$  is a transport coefficient associated with diffusion, and the column vector  $C$  and its transpose will be discussed below.

The dissipative contribution to the transport equations is given by (see [10], Eqs. (1.1) and (2.54))

$$M^{\text{diff}} \cdot \frac{\delta S}{\delta x} = M^{\text{diff}} \cdot \begin{pmatrix} -\frac{\mu}{T} \\ 0 \\ \frac{1}{T} \end{pmatrix}$$

where  $S$  denotes the entropy of the system, and  $\mu$  is the chemical potential;  $\frac{\delta}{\delta x}$  denotes a functional derivative. The friction matrix  $M^{\text{diff}}$  has to fulfill the requirement of degeneracy of energy  $E$  (see [10], Eqs. (1.5) and (2.49))

$$M^{\text{diff}} \cdot \frac{\delta E}{\delta x} = M^{\text{diff}} \cdot \begin{pmatrix} -\frac{1}{2}\mathbf{v}^2 \\ \mathbf{v} \\ 1 \end{pmatrix} = 0, \quad (2)$$

which requires  $C^T \cdot \frac{\delta E}{\delta x} = 0$ .

The elements of  $C$  are a vector, a tensor, and a vector. To construct  $C$  we assume that its elements must be tensorial combinations of  $\{\frac{\partial}{\partial \mathbf{r}}, \mathbf{v}, \bar{\mathbf{v}}\}$ , where  $\mathbf{v}$  denotes the velocity associated with momentum, and  $\bar{\mathbf{v}}$  denotes another velocity, which is introduced to account for possible differences between mass velocity and momentum velocity, as stated in the introduction.

The elements of the column vector  $C$  are subject to the following restrictions:

(a) The dissipative contribution to the transport equations,  $M^{\text{diff}} \cdot \frac{\delta S}{\delta x}$ , contains only second order space derivatives, so that the additional dissipative contributions have the same structure as the classical dissipative contributions from the laws of Navier-Stokes and Fourier; thus,  $C$  must be of first order in  $\frac{\partial}{\partial \mathbf{r}}$ .

(b) The contributions to mass and momentum balance must be of divergence form, so that these have the proper form of conservation laws; thus, the gradient must be on the left in the first and second term.

(c)  $C$  is at most quadratic in the velocities. Due to the degeneracy requirement (2) this implies that the first two elements of  $C$  must at most be linear in the velocities.

From the above follows immediately, that  $C^{(1)} = \frac{\partial}{\partial \mathbf{r}}$ , which implies  $C^{T(1)} = -\frac{\partial}{\partial \mathbf{r}}$ .

At first glance, the second element could be of the general form  $C^{(2)} = (\frac{\partial}{\partial \mathbf{r}} a_1 \bar{\mathbf{v}})^T + (\frac{\partial}{\partial \mathbf{r}} a_2 \mathbf{v})^T$ . We anticipate the requirements of Galilean invariance. Galilean invariance of the momentum balance is only fulfilled for  $a_1 + a_2 = 1$ , and since the velocity  $\bar{\mathbf{v}}$  is undefined yet, we can redefine as  $a_1 \bar{\mathbf{v}} + (1 - a_1) \mathbf{v} \rightarrow \bar{\mathbf{v}}$ . Thus, we have  $C^{(2)} = (\frac{\partial}{\partial \mathbf{r}} \bar{\mathbf{v}})^T$  and  $C^{T(2)} = -(\bar{\mathbf{v}} \frac{\partial}{\partial \mathbf{r}})^T$ .

The third component follows from the degeneration of energy as  $C^{T(3)} = \frac{\partial}{\partial \mathbf{r}} \frac{1}{2} \mathbf{v}^2 - \bar{\mathbf{v}} \cdot (\frac{\partial}{\partial \mathbf{r}} \mathbf{v})^T + \hat{\alpha} \frac{\partial}{\partial \mathbf{r}}$ . Here,  $\hat{\alpha}$  is a scalar quantity which possibly can depend on the velocities. Galilean invariance of the dissipative mass flux requires that  $(\hat{\alpha} - \frac{1}{2} \bar{\mathbf{v}}^2)$  is Galilei invariant,

so that we must have

$$\hat{\alpha} - \frac{1}{2}\bar{v}^2 = \bar{\alpha} + \beta(v - \bar{v})^2 = \alpha$$

where  $\bar{\alpha}$ ,  $\beta$  are velocity independent scalars, and  $\alpha$  is a velocity dependent Galilei invariant scalar.

There actually is the possibility to add a further term of the form  $(\partial/\partial\mathbf{r})(\hat{\mathbf{v}} - \mathbf{v})$  to  $C^{(2)}$ , without transposition, and a compensating term  $(\hat{\mathbf{v}} - \mathbf{v})\text{div}\mathbf{v}$  to  $C^{(3)}$ . This ansatz involves a further velocity field  $\hat{\mathbf{v}}$ . The resulting additional pressure tensor contribution turns out to be isotropic and proportional to  $\text{div}\mathbf{v}$ . In the following, we do not further consider such a bulk-viscosity type effect by choosing  $\hat{\mathbf{v}} = \mathbf{v}$ .

Summarizing, we have the following

$$C = \begin{pmatrix} \frac{\partial}{\partial\mathbf{r}} \\ (\frac{\partial}{\partial\mathbf{r}}\bar{\mathbf{v}})^T \\ \frac{1}{2}\mathbf{v}^2\frac{\partial}{\partial\mathbf{r}} - (\mathbf{v}\frac{\partial}{\partial\mathbf{r}})^T \cdot \bar{\mathbf{v}} + \frac{\partial}{\partial\mathbf{r}}(\frac{1}{2}\bar{\mathbf{v}}^2 + \alpha) \end{pmatrix}, \quad (3)$$

and

$$C^T = - \begin{pmatrix} \frac{\partial}{\partial\mathbf{r}} & (\bar{\mathbf{v}}\frac{\partial}{\partial\mathbf{r}})^T & \frac{\partial}{\partial\mathbf{r}}\frac{1}{2}\mathbf{v}^2 - \bar{\mathbf{v}} \cdot (\frac{\partial}{\partial\mathbf{r}}\mathbf{v})^T + (\frac{1}{2}\bar{\mathbf{v}}^2 + \alpha)\frac{\partial}{\partial\mathbf{r}} \end{pmatrix}. \quad (4)$$

In these equations, the velocity field  $\mathbf{v} = \mathbf{M}/\rho$  occurs as the functional derivative of the energy with respect to the momentum density field, whereas  $\bar{\mathbf{v}}$  and  $\alpha$  appear as free parameters in the most general form of the friction matrix and need to be interpreted according to their respective role in the hydrodynamic equations. For  $\bar{\mathbf{v}} = \mathbf{v}$ , one recovers the previously suggested scenario formulated in [10]. The parameter  $\alpha$  implies a cross effect describing the influence of a temperature gradient on the diffusive mass flux. We refer to  $\bar{\mathbf{v}}$  as a smoothed velocity field. As will be seen in the course of our arguments, non-vanishing  $\bar{\mathbf{v}}$  is necessary to obtain all required properties of the modified hydrodynamic equations. The question whether such a velocity exists will be discussed in our conclusions (Section VIII) based on the various restrictions that will become apparent in the course of the paper.

The generalized contribution  $M^{\text{diff}}$  defined in Eqs. (1)–(4) has all the properties required by the thermodynamic framework: by construction  $M^{\text{diff}}$  is symmetric, positive-semidefinite, and fulfills the degeneracy of energy. The rank of  $M^{\text{diff}}$  is chosen to be unity because  $M^{\text{diff}}$  represents a single additional dissipative process.

For the generalized friction matrix with the contribution (1) due to diffusion, after transforming from the variables  $x$  to  $x'$ , we obtain a hydrodynamic scenario expressed by the

following equations:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{M} - \mathbf{j}\rho), \quad (5)$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left( \frac{1}{\rho} \mathbf{M} \mathbf{M} - \mathbf{j}\rho \bar{\mathbf{v}} + p\mathbf{1} + \boldsymbol{\tau} \right), \quad (6)$$

and

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left( \frac{1}{\rho} \mathbf{M} s + \mathbf{j}^s \right) + \frac{\rho^2 \mathbf{j}^2}{D'} + \frac{\mathbf{j}^q{}^2}{\lambda^q T^2} + \frac{\boldsymbol{\tau} : \boldsymbol{\tau}}{2\eta T}. \quad (7)$$

For completeness, we also give the energy balance, which can be obtained from combining mass and entropy balance via the Gibbs equation,

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left( \frac{\mathbf{M}}{\rho} u + \mathbf{j}^e \right) + (-p\mathbf{1} + \boldsymbol{\tau}) : \frac{\boldsymbol{\tau}}{2\eta} + \rho(\bar{\mathbf{v}} - \mathbf{v}) \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \cdot \mathbf{j}. \quad (8)$$

On the right-hand sides of these equations, there occur the local equilibrium pressure  $p = p(\rho, s)$ , as well as the dissipative fluxes of mass ( $\rho \mathbf{j}$ ), momentum ( $\boldsymbol{\tau}$ ), heat ( $\mathbf{j}^q$ ), entropy ( $\mathbf{j}^s$ ), and energy ( $\mathbf{j}^e$ ).

These fluxes are given in terms of transport coefficients and gradients of intensive quantities:

$$\mathbf{j}^q = \lambda^q T^2 \frac{\partial}{\partial \mathbf{r}} \frac{1}{T} = -\lambda^q \frac{\partial T}{\partial \mathbf{r}} \quad (9)$$

is the thermal contribution to non-convective energy transport, in terms of the thermal-conductivity parameter  $\lambda^q$  and the gradient of the absolute temperature  $T$ . The complete non-convective energy flux is given by

$$\mathbf{j}^e = \mathbf{j}^q - \rho \mathbf{j} \left( \alpha + \frac{1}{2} (v - \bar{v})^2 \right);$$

clearly,  $\alpha$  describes a cross effect. Furthermore,

$$\boldsymbol{\tau} = -\eta \left[ \frac{\partial}{\partial \mathbf{r}} \mathbf{v} + \left( \frac{\partial}{\partial \mathbf{r}} \mathbf{v} \right)^T - \frac{2}{3} \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{v} \mathbf{1} \right] \quad (10)$$

is Newton's expression for the stress tensor in terms of the viscosity  $\eta$  and the gradient of the velocity (as mentioned before, it is convenient to assume that the bulk viscosity vanishes), and

$$\rho \mathbf{j} = D' \left[ \frac{\partial}{\partial \mathbf{r}} \frac{\tilde{\mu}}{T} - \alpha \frac{\partial}{\partial \mathbf{r}} \frac{1}{T} - \frac{1}{T} (\mathbf{v} - \bar{\mathbf{v}}) \cdot \left( \frac{\partial}{\partial \mathbf{r}} \bar{\mathbf{v}} \right)^T \right] \quad (11)$$

is the dissipative mass flux in terms of the coefficient  $D'$ , which is closely related to the diffusion coefficient, the cross coefficient  $\alpha$ , and the gradient of a velocity-modified version of the chemical potential per unit mass  $\mu$  in addition to the gradient of temperature

$$\tilde{\mu} = \mu - \frac{1}{2} (\mathbf{v} - \bar{\mathbf{v}})^2; \quad (12)$$

the total mass flux is given by  $(\mathbf{M} - \mathbf{j}\rho)$ .

Finally,

$$\mathbf{j}^s = \frac{1}{T}\mathbf{j}^q + \frac{\tilde{\mu} - \alpha}{T}\rho\mathbf{j} = \frac{1}{T}\mathbf{j}^e + \frac{\mu}{T}\rho\mathbf{j} \quad (13)$$

is the total entropy flux; the second equation shows the expected form of the entropy flux for systems with energy and diffusion fluxes.

There occurs a natural quadratic entropy production term associated with the dissipative mass flux  $\rho\mathbf{j}$  (that is, with diffusion), so that the entropy production remains positive semidefinite. Equation (7) suggests that the added diffusion effect is on the same footing as the dissipative effects in conventional hydrodynamics.

The natural occurrence of a velocity-dependent chemical potential has previously been found in [15] (however, without  $\bar{\mathbf{v}}$ ). Such a modification of the chemical potential suggests that a possible diffusion process is more subtle than viscous stresses or thermal conductivity. Note that the generalizations occur only through second-order terms in the deviations from equilibrium. In linearized hydrodynamics,  $\bar{\mathbf{v}}$  does not play any role.

From a conceptual perspective, it is important to notice that we need to introduce two different velocities  $\mathbf{v}$  and  $\bar{\mathbf{v}}$  before we can formulate a dissipative contribution to the mass flux. Therefore, the discrepancy between mass flux and momentum density is not the primary reason for the occurrence of two different velocities; rather two different velocities are needed to introduce diffusion into hydrodynamics. Unlike momentum velocity  $\mathbf{v} = \frac{\mathbf{M}}{\rho}$  and mass velocity  $\frac{\mathbf{M}}{\rho} - \mathbf{j}$ , the additional velocity  $\bar{\mathbf{v}}$  does not have an obvious intuitive interpretation; as the equations show, non-zero  $\bar{\mathbf{v}}$  would manifest itself through its contribution to diffusion.

### III. GALILEAN INVARIANCE

Galilean invariance was already used in the construction of the friction matrix  $M^{\text{diff}}$ , although we did not present explicit arguments to shorten the presentation.

If we introduce  $\mathbf{v}_0$  as an arbitrary constant velocity shift, and make the replacements

$$\mathbf{M} \rightarrow \mathbf{M} + \rho\mathbf{v}_0, \quad \bar{\mathbf{v}} \rightarrow \bar{\mathbf{v}} + \mathbf{v}_0, \quad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \mathbf{v}_0 \cdot \frac{\partial}{\partial \mathbf{r}}, \quad (14)$$

in the hydrodynamic scenario given by the mass balance (5), the momentum balance (6), and the entropy balance (7), they keep their original form. For this to be true it is essential

that  $\mu$  and  $\tilde{\mu}$  and hence also  $\mathbf{j}$  and  $\mathbf{j}^s$  are invariant under Galilean transformations, and also the cross coefficient  $\alpha$  needs to be invariant. As indicated in the previous section, the requirement of invariance already was considered in the construction of the friction matrix. We note that invariance of the transport equations does not require invariance of the friction matrix.

While Galilean invariance concerns the transformation between inertial systems, one can also consider the question of transformations into non-inertial frames, which in classical hydrodynamics leads to the occurrence of inertial forces [20]. In the present context of modified hydrodynamics with diffusive mass flow, there will be inertial terms in the mass flux as well, with subtle consequences for material objectivity [20]. Our line of arguments does not require discussion of non-inertial frames, and will lead to the conclusion that diffusive contributions to mass flow are not possible; hence we refrain from a more detailed discussion of these issues.

#### IV. INTEGRABILITY

Subtracting  $\left(-\frac{\partial \mathbf{j}\rho}{\partial t}\right)$  from both sides of the momentum balance (6) gives the transport equation for the total mass flux  $(\mathbf{M} - \mathbf{j}\rho)$ . This equation is a conservation law if a suitable function  $J$  exists as a potential for  $\mathbf{j}\rho$ , so that

$$\mathbf{j}\rho = \frac{\partial J}{\partial \mathbf{r}}. \quad (15)$$

In general,  $J$  could be a tensor, but in the following we consider only scalar functions. Note that the existence of a potential  $J$  is sufficient for local conservation of the mass flux, but not necessary, that is, the mass flux could be conserved under weaker conditions. For the further arguments, however, a detailed discussion of the condition (15) turns out to be useful.

For constant (i.e., homogeneous) parameters  $D'$ ,  $\alpha$ , and  $\bar{\mathbf{v}}$  in Eqs. (11), the dissipative contribution to the mass flux can trivially be written in the form (15). However, such a representation holds under much weaker conditions on  $D'$ ,  $\alpha$ , and  $\bar{\mathbf{v}}$ , and that is our motivation for proposing it. In the following, we refer to Eq. (15) as the *integrability condition*.

A local conservation law for the mass flux is, in fact, expected from general considerations of statistical mechanics [19], and it probably is the reason why one usually concludes without hesitation that the mass flux must coincide with the well-known conserved momentum

density. Below it will be shown that a second conservation law originates in the condition (15) which, in turn, is needed to obtain a uniform center-of-mass motion.

We consider the most general form for  $J$  as a function of the hydrodynamic variables up to second order terms in velocity, which reads

$$J = J_1(\mu, T) - \frac{1}{2} J_2(\mu, T)(\mathbf{v} - \mathbf{c})^2, \quad (16)$$

Galilean invariance of  $J$  requires the dependence on a velocity difference, hence the appearance of the additional velocity  $\mathbf{c}$ , which will be determined by comparing Eqs. (11) and (15) for the ansatz (16). We find right away that

$$\frac{\partial \mathbf{c}}{\partial \mathbf{r}} = 0, \quad (17)$$

that is the velocity  $\mathbf{c}$  must be homogeneous (constant in space).

Next, we identify the diffusion coefficient as

$$\frac{D'}{T} = \frac{\partial J_1}{\partial \mu} - \frac{1}{2} \frac{\partial J_2}{\partial \mu} (\mathbf{v} - \mathbf{c})^2. \quad (18)$$

Thus, an arbitrary functional form of the diffusion coefficient can be reproduced by choosing a suitable  $\mu$ -dependence of  $J_1$  and  $J_2$ . In fact, to restrict the constitutive law (11) to second order in velocity, one will choose  $\frac{\partial J_2}{\partial \mu} = 0$ , so that  $J_2 = J_2(T)$ .

By comparing the remaining terms, we further obtain

$$\bar{\mathbf{v}} = \mathbf{v} + \frac{T J_2(T)}{D'} (\mathbf{c} - \mathbf{v}), \quad (19)$$

and

$$\alpha = \mu + \frac{T^2}{D'} \frac{\partial J_1}{\partial T} - \frac{1}{2} \left( 1 - D' \frac{\partial}{\partial T} \frac{1}{J_2} \right) (\mathbf{v} - \bar{\mathbf{v}})^2. \quad (20)$$

We consider special choices of  $J_2$ , and their influence on the constitutive equation (19) for  $\bar{\mathbf{v}}$ . Note that the previously suggested choice  $J_2 = 0$  [10] gives  $\bar{\mathbf{v}} = \mathbf{v}$  by Eq. (19) in agreement with [10]. If  $D'$  depends only on temperature, the natural choice  $J_2 = D'(T)/T$  leads to the simple expression  $\bar{\mathbf{v}} = \mathbf{c}$ ; in this case, because of (17),  $\bar{\mathbf{v}}$  must be a constant. However, if  $D'$  depends also on  $\mu$ , the reference velocity  $\bar{\mathbf{v}}$  cannot just be a constant.

With the assumption that for an ideal gas the mass diffusion coefficient  $D'$  behaves similar to the self-diffusion coefficient,  $D' \propto \sqrt{T}$  depends only on  $T$ , so that  $J_1 = \mu D'/T$  and  $J_2 = D'/T$  are the most natural choices. We hence find

$$\bar{\mathbf{v}} = \mathbf{c}, \quad \alpha = \frac{1}{2} \tilde{\mu}. \quad (21)$$

The example of the ideal gas demonstrates that the integrability condition can indeed be consistent with the generalized hydrodynamic scenario. The expression (21) for  $\alpha$  differs from the values considered in the pertinent literature [4, 10]. While one might argue that the above result for  $\alpha$  for an ideal gas casts the idea of integrability into doubt, one should take into account that all suggestions for non-zero values of  $\alpha$  and  $D'$  (and thus non-zero dissipative contribution to mass flux) are based solely on plausibility arguments, but not on derivation from first principles nor on experimental evidence. Note also that, under the assumption of integrability, the above discussion suggests that  $\bar{\mathbf{v}}$  is homogeneous (at least for the ideal gas)!

## V. ANGULAR MOMENTUM

In any hydrodynamic theory, one should be able to construct a locally conserved angular momentum resulting from the existence of rotational symmetry [21]. The conservation or non-conservation of angular momentum is related to the structure of the momentum balance equation. When the latter is taken into the cross product with  $\mathbf{r}$ , we find a balance for the local angular momentum density  $\mathbf{l} = \mathbf{r} \times \mathbf{M}$ , which we write in index notation as

$$\frac{\partial l_i}{\partial t} = -\frac{\partial}{\partial r_l} \left[ \frac{1}{\rho} M_l l_i + \epsilon_{ijk} r_j (-j_l \rho \bar{v}_k + p \delta_{kl} + \tau_{kl}) \right] - \epsilon_{ijk} \rho j_j \bar{v}_k. \quad (22)$$

One finds  $-\epsilon_{ijk} \rho j_j \bar{v}_k = (\bar{\mathbf{v}} \times \mathbf{j}\rho)_i$  as a local source of angular momentum. This contribution stems from the non-symmetric contribution  $-\mathbf{j}\rho\bar{\mathbf{v}}$  to the momentum flux tensor, which therefore potentially causes a problem. With the identity

$$\epsilon_{ijk} \rho j_j \bar{v}_k = \frac{\partial}{\partial r_l} [\epsilon_{ljk} \rho j_j \bar{v}_k r_i] + r_i \bar{v}_k \left( \epsilon_{kjl} \frac{\partial \rho j_j}{\partial r_l} \right) - r_i \rho j_j \left( \epsilon_{jkl} \frac{\partial \bar{v}_k}{\partial r_l} \right), \quad (23)$$

it becomes clear that we can obtain a locally conserved angular momentum provided that the last two terms in (23) vanish, that is if (i) the integrability condition (15) holds, and (ii) the modified velocity field  $\bar{\mathbf{v}}$  is curl-free.

Clearly, condition (ii) is fulfilled if we take  $\bar{\mathbf{v}}$  as independent of position. For the previous choice,  $\bar{\mathbf{v}} = \mathbf{v}$ , however, this condition will not be fulfilled, that is this choice would lead to violation of conservation of angular momentum, and thus is not physical!

Conservation of angular momentum together with the integrability requirement of Section IV leads to a strong reduction of the possible constitutive equations: Together with curl-free

$\bar{\mathbf{v}}$  the condition (19) can only be fulfilled for arbitrary fields  $T, \mu, \mathbf{v}$  if  $D'$  is independent of  $(\mathbf{v} - \mathbf{c})$ . Because of (18), this implies that  $J_2$  must be independent of  $\mu$ , and then vanishing curl of (19) requires  $D' = T J_2(T)$ , so that  $D'$  must depend only on  $T$ . With this, (18) reduces to  $D'(T) = T \frac{\partial J_1}{\partial \mu}$  which implies  $J_1(\mu, T) = \frac{\mu}{T} D'(T)$ . Thus, integrability and conservation of angular momentum together require  $\bar{\mathbf{v}} = \mathbf{c}$ , that is constant  $\bar{\mathbf{v}}$ . We found the same in Section IV for the ideal gas; the present argument extends the findings to all materials as sufficient, but not necessary conditions.

Can one construct a locally conserved angular momentum without assuming integrability? It is generally expected that a non-symmetric stress tensor is associated with some internal angular momentum. However, we did not succeed in constructing an internal angular momentum in such a way that the total angular momentum is conserved. A hydrodynamic theory without the possibility to construct a locally conserved angular momentum density field needs to be rejected. The only construction we could find is based on integrability and constant  $\bar{\mathbf{v}}$  and hence rules out the previous formulations of modified hydrodynamics [4, 10] with a dissipative contribution to the mass flux based on  $\bar{\mathbf{v}} = \mathbf{v}$ . Any alternative suggestion of a scenario with a dissipative mass flux must come with an explicit suggestion for the total angular momentum density.

## VI. STEADY RIGID FLUID ROTATION

To obtain a further criterion for the admissibility of a dissipative contribution to the mass flux, we look for a particular solution of the different sets of hydrodynamic equations introduced in the preceding sections. Namely, we expect that a fluid performing a rigid body rotation should provide a solution of any valid set of hydrodynamic equations. The existence of dissipation free rigid rotations is not a necessary condition. If one believes that such solutions should exist, however, one finds additional restrictions on modifications of transport processes.

In particular, for steady rigid fluid rotation, all dissipative flux contributions must vanish. Moreover, the pressure must increase radially to provide the required centripetal forces,

$$\frac{dp}{dr} = \rho \frac{v^2}{r}, \quad (24)$$

where  $r$  is the distance from the axis of rotation.

For the further arguments, it is important that thermodynamic local equilibrium states are fully characterized by two intensive state variables. Because the pressure  $p$  must have the spatial variation (24), at most one intensive local-equilibrium variable can be constant throughout the system.

We first consider the case where  $\bar{\mathbf{v}} = \mathbf{v}$ , as was suggested for the original modifications to hydrodynamics [4, 10]. For this case, according to Eqs. (9) and (11), vanishing dissipative fluxes  $\mathbf{j}^q = \mathbf{j} = \mathbf{0}$  would require that the two intensive variables  $T$  and  $\mu$  be constant throughout the system, which is at variance with the spatial variation of  $p$  given in Eq. (24). This implies that, for the previously proposed formulation, there can only be a single process, either heat conduction (that is, standard hydrodynamics), or just the combined effect of diffusion and heat conduction described by  $\mathbf{j}$ . In other words, two dissipative processes driven by gradients of two intensive local-equilibrium variables, as assumed in all previous papers [2, 4, 9, 10, 11, 12], are excluded by the assumption of a solution describing steady rigid fluid rotation.

For a single combined Brenner-type process, one expects to find a nontrivial temperature profile for the rigidly rotating fluid. This is so, because the pressure profile is dictated by the centripetal forces, and the combination of pressure and temperature gradients that drives the mixed process must vanish in the dissipation-free state of rotation. Note, however, that the existence of such a solution for a rigidly rotating fluid does not guarantee the general conservation of angular momentum.

For our more general formulation in the center-of-mass system, for which we assume  $\bar{\mathbf{v}} = \mathbf{0}$ , according to Eqs. (9), (11) and (12), vanishing dissipative fluxes  $\mathbf{j}^q = \mathbf{j} = \mathbf{0}$  would require only the intensive variable  $T$  to be constant throughout the system, whereas constant  $\tilde{\mu}$  for rigid rotation translates into

$$\frac{d\mu}{dr} = \frac{d}{dr} \left( \frac{1}{2} \mathbf{v}^2 \right) = \frac{\mathbf{v}^2}{r}. \quad (25)$$

According to the Gibbs-Duhem relation, that is,  $dp = sdT + \rho d\mu$ , Eq. (25) is perfectly consistent with the radial pressure distribution (24) at constant temperature.

Whereas inhomogeneous equilibria caused by external forces can be described in the previously suggested formulation because the external forces appear in the thermodynamic driving forces of the dissipative processes [11], the inhomogeneous equilibrium due to fluid rotation can only be handled through the velocity-modified chemical potential of the gener-

alized approach.

Note, however, that the above argument requires  $\bar{\mathbf{v}} = \mathbf{0}$  in the center-of-mass system, this condition must be seen as a severe constriction for  $\bar{\mathbf{v}}$ . Integrability has not been assumed in this section.

## VII. CENTER-OF-MASS POSITION

In the absence of external forces, the center-of-mass motion of a system must be uniform. This fundamental insight allows us to reconstruct the center-of-mass position at time  $t = 0$ , which is a constant of motion. More generally, one actually expects

$$\mathbf{b} = \rho \mathbf{r} - M \mathbf{t} \quad (26)$$

to be the density of a locally conserved quantity. From the perspective of special relativity, this locally conserved quantity  $\mathbf{b}$  is the natural partner of the angular momentum density  $\mathbf{l}$ , and  $\mathbf{b}$  is known as the booster density [15]. From the definition (26) and the balance equations (5), (6), we obtain

$$\frac{\partial \mathbf{b}}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left[ \frac{1}{\rho} M \mathbf{b} - \mathbf{j} \rho (\mathbf{r} - \bar{\mathbf{v}} t) - t(p \mathbf{1} + \boldsymbol{\tau}) \right] - \mathbf{j} \rho. \quad (27)$$

Under the integrability condition (15), the uniform center-of-mass motion is associated with a locally conserved quantity  $\mathbf{b}$ . Integrability, locally conserved angular momentum, locally conserved mass flux, and locally conserved booster are hence intimately related features. The generalized setting introduced in this paper provides a formally acceptable modification of standard Navier-Stokes-Fourier hydrodynamics, provided that we assume integrability.

Like for the angular momentum density  $\mathbf{l}$ , we might consider the possibility to define the booster density  $\mathbf{b}$  in terms of the mass flux instead of the momentum density. Such a definition would seem plausible because the local effects on the rearrangement of the center-of-mass position are given by the mass flux. It then becomes evident that a local conservation law for the modified booster is directly equivalent to a local conservation law for the mass flux, without using integrability as an intermediate link. There hence exists a deeper reason for a separate conservation law for the mass flux.

## VIII. CONCLUSIONS AND OPEN PROBLEMS

In spite of its thermodynamic admissibility, the most widely discussed scenario for introducing a dissipative contribution to the mass flux in hydrodynamics [2, 4, 9, 10, 11, 12] must be rejected because it violates the local conservation of angular momentum. Moreover, it does not allow for steady rigid fluid rotation. Through the occurrence of an additional curl-free velocity  $\bar{\mathbf{v}}$ , the balance equations (5)–(7) together with the constitutive equations (9)–(11) provide a new generalized set of hydrodynamic equations with a dissipative contribution to the mass flux. For a suitable choice of the smoothed velocity field  $\bar{\mathbf{v}}$ , and with the assumption of an integrability condition, all the following criteria can be satisfied:

- Galilean invariance,
- possibility of steady rigid fluid rotation,
- existence of a locally conserved angular momentum,
- and consistency with uniform center-of-mass motion.

In addition to the momentum density, also the mass flux turns out to be a locally conserved quantity. The deeper reason for this conservation of the mass flux lies in the existence of a locally conserved booster density. In the new generalized formulation, an integrability condition ensures the existence of the proper conservation laws.

Although it satisfies all the fundamental admissibility criteria, the new generalized formulation of hydrodynamics must be scrutinized in view of the role of the additional velocity field  $\bar{\mathbf{v}}$ . Assuming integrability, we found that  $\bar{\mathbf{v}}$  must be curl-free to satisfy conservation of angular momentum (Section V), that it must vanish in the center of mass of a rotating system to allow rigid fluid rotation (Section VI), and that it must be constant to guarantee conservation of mass flux (Sections IV/VI).

From this list, one might be inclined to assume that  $\bar{\mathbf{v}}$  is independent of position, that is, homogeneous, but possibly time-dependent, and that the center-of-mass velocity is the simplest interpretation of  $\bar{\mathbf{v}}$ . However, this leads immediately to the question of how to define the system for the determination of the center of mass. While for certain systems a particular choice of boundaries might be self-suggesting, for example the walls of a vessel, a

local continuum theory must be valid for any choice of system (provided that the system is bigger than a meaningful “point” of the continuum).

As we look for a local theory of hydrodynamics, any given system can be divided into smaller subsystems in an arbitrary manner, and this division should not alter the mathematical description of the physics within the system, or the subsystems. Thus, if  $\bar{\mathbf{v}}$  is a homogeneous velocity in a system  $A$ , it must be the same velocity in any subsystem of  $A$ . Also, if  $A$  is combined with another system  $B$ , the combined system  $A + B$  must still have the same  $\bar{\mathbf{v}}$ , since else the local description in  $A$  would be altered. As a logical consequence follows that  $\bar{\mathbf{v}}$  would have to be a universal constant. This is inconsistent with Galilean invariance.

Only when the dissipative mass flux vanishes ( $D' = 0$ ) do we have no contradiction—but this is just the case of classical hydrodynamics. Thus, we come to the conclusion that a dissipative mass flux necessarily leads to contradiction to basic physical principles, and must be rejected. It follows that classical hydrodynamics, where mass flux and momentum density agree, is the only admissible system of transport equations in the hydrodynamic regime (i.e., at small Knudsen numbers; for finite Knudsen numbers rarefaction effects must be accounted for which leads to more complicated forms of the transport equations which all reduce to classical hydrodynamics in the limit of sufficiently small Knudsen numbers [7]).

A particular example arises by generalizing our example of steady rigid fluid rotation. If a system consists of two independent, rigidly rotating subsystems in relative motion, the two individual center-of-mass velocities must be used to obtain a proper overall solution. The averaging is no longer global but only over the subsystems. If, in general, the averaging required to obtain the smoothed velocity field cannot be global, then the question about the proper spatial averaging procedure arises. We are then faced with mathematical and physical issues, and we here focus on the latter, in particular: What is the relevant physical length scale for smoothing?

The fundamental problem associated with a dissipative contribution to the mass flux is that it represents random fluctuations in the position of non-inertial particles and hence destroys the uniform center-of-mass motion implied by Newton’s equations of motion. Only after some averaging procedure one can expect a smooth center-of-mass motion. This averaging should at least comprise several collisions, which define the intrinsic physical scale of hydrodynamic equations. The collisions also define the characteristic scales needed to reach

local equilibrium. As the velocity  $\mathbf{v}$  of classical hydrodynamics should clearly be defined on the local-equilibrium scale, there is no natural way of introducing a further smoothed velocity  $\bar{\mathbf{v}}$  based on intrinsic length scales. Only if the hydrodynamic velocity  $\mathbf{v}$  is assumed to be well-defined on a more local scale [22], as implicitly done in fluctuating hydrodynamics (see Chapter IX of Landau and Lifshitz [23] or Section 2.2.4 of [10]) or in fluctuation renormalization (related to eliminating the back-flow effects of hydrodynamics leading to long-time tails as described in Chapter 16 of [24]), then there exists a natural smoothing procedure. However, we then enter the domain of finite Knudsen number effects and hence leave the scope of the purely hydrodynamic theory discussed in this paper.

While a dissipative contribution to the mass flux seems to be irrelevant to continuum hydrodynamics, a deeper understanding of the related phenomena may still be useful to resolve some problems arising in the theory of fluctuations [14]. As a consequence, Einstein's theory of local equilibrium fluctuations can only be valid on length scales large compared to the mean free path, even for rarefied gases. More disturbingly, the use of Onsager's regression hypothesis [24, 25] to evaluate the two-time correlations determining the dissipative properties of nonequilibrium systems according to the Green-Kubo formula [10, 24, 26] needs to be questioned. However, it might also happen that the problems related to the description of fluctuations in hydrodynamics can only be resolved in a more complete consistent theory at finite Knudsen numbers.

To summarize, we state that all known modifications of hydrodynamics that lead to a dissipative contribution to mass flux violate physical principles and must be discarded.

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